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SPECTRAL MIMICRY: A METHOD OF SYNTHESIZING MATCHING TIME SERIES WITH DIFFERENT FOURIER SPECTRA*

Joel E. Cohen,[1,2,3] *Charles M. Newman,*[4]
Adam E. Cohen,[3,5] *Owen L. Petchey,*[3]
and Andrew Gonzalez[3]

Abstract. Given a stationary time series X and another stationary time series Y (with a different power spectral density), we describe an algorithm for constructing a stationary time series Z that contains exactly the same values as X permuted in an order such that the power spectral density of Z closely resembles that of Y. We call this method *spectral mimicry*. We prove (under certain restrictions) that, if the univariate cumulative distribution function (CDF) of X is identical to the CDF of Y, then the power spectral density of Z equals the power spectral density of Y. We also show, for a class of examples, that when the CDFs of X and Y differ modestly, the power spectral density of Z closely approximates the power spectral density of Y. The algorithm, developed to design an experiment in microbial population dynamics, has a variety of other applications.

1. Introduction

Suppose that we are given two scalar-valued time series X and Y of equal finite length, i.e., two samples of the same finite duration from different stationary stochastic processes on the natural numbers. (For an accessible introduction to the concepts and terminology of stationary time series, see, e.g., [4, Chapter 47].) We describe here a method of constructing a third time series Z that contains exactly

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[1]Rockefeller University, 1230 York Avenue, New York, New York 10021-6399.
E-mail: cohen@rockvax.rockefeller.edu
[2]Columbia University, New York, New York 10027.
[3]Centre for Population Biology, Imperial College at Silwood Park, Ascot, Berks, SL5 7PY, United Kingdom.
[4]Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012.
[5]Harvard University, Cambridge, Massachusetts 02138.

the same elements as X permuted so that the power spectral density of Z resembles the power spectral density of Y. We call this method *spectral mimicry*. We shall prove (under certain restrictions) that, if the univariate cumulative distribution function (CDF) of the stochastic process that generates X is identical to the CDF of the stochastic process that generates Y, then the power spectral density of Z equals the power spectral density of Y. We also show, for a class of examples, that when the CDFs of X and Y differ modestly, the power spectral density of Z closely approximates the power spectral density of Y.

The development of spectral mimicry was motivated by the need to design an experiment in microbial population dynamics [2]. Ecological theorists have predicted that the relative importance of high-frequency versus low-frequency fluctuations in environmental features such as temperature will affect the composition and dynamics of ecological communities [6]–[8], [1]. However, the impact of the shape of the power spectrum of an environmental parameter on the population dynamics of individual species remained to be investigated experimentally. If one population of microbes were grown in an environment where the temperature fluctuated like a white noise (i.e., with a power level independent of frequency), and another population experienced temperatures that fluctuated like a red noise (i.e., with a power level that decreases with increasing frequency), how would the population dynamics of the microbes be affected?

The experiment required the construction of two types of times series of temperatures — one type approximating white noise and the other approximating red noise — which were otherwise as similar as possible. Generating white noise (the series X described in the first paragraph) is easily done by sampling from a pseudorandom number generator. Generating red noise (the series Y of the first paragraph) is easily done by generating the simplest nontrivial autoregressive series, a Markov-Gaussian process [4, Example 47.7, p. 418] with positive autocorrelation. The method of spectral mimicry then makes it possible to generate a red noise having exactly the values of X and the power spectral density of Y. We shall prove these assertions and give a numerical example.

2. The method

We denote time series data of finite positive length T by $X = (X(1), X(2), \ldots, X(T))$, where the elements occur in the order $X(1), X(2), \ldots,$ up to $X(T)$. We denote by $\{X(t)\}$ the sequence of elements of X without regard to temporal order, and by $X[1] \leq X[2] \leq \cdots \leq X[T]$ the T order statistics of X, that is, the elements of X rearranged in nondecreasing order. For example, $X[1] = \min\{X(t)\}$ and $X[T] = \max\{X(t)\}$. We use identical notation for the other time series Y and Z.

We assume that the data X are generated by a stationary sequence of random variables with finite mean and variance. We assume that an equally long sequence of data Y are generated by another stationary sequence of other random variables with finite mean and variance, and that $\{X(t)\}$ differs from $\{Y(t)\}$. We aim to construct

a sequence Z that has exactly the same values as X, i.e., $\{X(t)\} = \{Z(t)\}$, such that, asymptotically as T becomes large, the time series Z has a power spectrum with the same relative shape (i.e., increasing or decreasing with frequency) as that of Y.

To achieve this, we construct Z from Y by replacing each element of Y by the corresponding *order statistic* of X. That is, if $Y(t) = Y[s]$, we define $Z(t) = X[s]$, for $t = 1, \ldots, T$. It is then obvious that $\{X(t)\} = \{Z(t)\}$, i.e., the elements of X and Z are identical apart from order.

3. Exact results

Recall that we use the term *stochastic process* to refer to a sequence of random variables, and the term *time series* to refer to a data sequence (e.g., one sampled from some stochastic process). For any set $\{\cdot\}$, let $\#\{\cdot\}$ denote the number of elements in the set $\{\cdot\}$. Let $X(1), X(2), \ldots$ and $Y(1), Y(2), \ldots$ be time series (of infinite length) with the following property: There exist functions $F(x)$ and $G(y)$, where F and G are nondecreasing functions defined on the entire real line and taking values in $[0, 1]$, such that as $T \to \infty$,

$$\text{for every real } x, \quad \frac{1}{T}\#\{t : 1 \le t \le T \text{ and } X(t) \le x\} \to F(x)\,, \tag{3.1}$$

$$\text{for every real } y, \quad \frac{1}{T}\#\{t : 1 \le t \le T \text{ and } Y(t) \le y\} \to G(y)\,. \tag{3.2}$$

Assume that $Y(1), Y(2), \ldots$ are all distinct and define $S_1^T, S_2^T, \ldots, S_T^T$ (depending on $Y(1), \ldots, Y(T)$) so that $Y(t) = Y\left[S_t^T\right]$ for $t = 1, 2, \ldots, T$ (where $Y[S]$ are the order statistics for Y). Define $Z^T(t)$ for $1 \le t \le T$ by $Z^T(t) = X\left[S_t^T\right]$. (This is the method of spectral mimicry.) We do not assume that $X(1), X(2), \ldots$ are all distinct.

When F is a strictly increasing function (i.e., if $F(x_1) < F(x_2)$ whenever $x_1 < x_2$), we define a function $F^{-1}(u)$ on $[0, 1]$ as follows. First denote by $F(x-)$ and $F(x+)$, respectively, the limits from the left and the right of F at x and denote by $F(-\infty)$ and $F(+\infty)$ the limits as $x \to -\infty$ and $x \to +\infty$ of F. Then define

$$F^{-1}(u) = \begin{cases} \text{the (unique) } x \text{ such that} \\ F(x-) \le u \le F(x+), & \text{if } F(-\infty) < x < F(+\infty), \\ -\infty, & \text{if } x \le F(-\infty), \\ +\infty, & \text{if } x \ge F(+\infty). \end{cases} \tag{3.3}$$

If F is continuous as well as strictly increasing (as is the standard normal distribution function), then F^{-1} is just the inverse function of F.

Theorem 1. *If F is strictly increasing, then for any fixed t_0, as $T \to \infty$,*

$$Z^T(t_0) \to \tilde{Y}(t_0) \equiv F^{-1}\left(G(Y(t_0))\right)\,, \tag{3.4}$$

where F^{-1} is defined by (3.3).

Proof. It follows directly from (3.2) (with $y = Y(t_0)$) that

$$\lim_{T\to\infty} S_{t_0}^T/T = G(Y(t_0)) \ . \tag{3.5}$$

It similarly follows from (3.1) that if $u \in [0, 1]$ and W_T takes values in $\{1, 2, \ldots, T\}$, then

$$W_T/T \to u \text{ implies } X[W_T] \to F^{-1}(u) \ . \tag{3.6}$$

Taking $W_T = S_{t_0}^T$ and combining (3.5) and (3.6) yield (3.4). □

If X and Y are stationary stochastic processes and (3.1) and (3.2) are valid with probability 1 (for which we henceforth write w.p.1), then $F(x) = P\{X(t') \le x\}$ with $F(-\infty) = 0$, $F(+\infty) = 1$ (F is the univariate CDF of X), and similar formulae apply to G and Y. This will be the case for any ergodic stationary process, including, for example, Gaussian stochastic processes, where the CDF is a normal distribution function. Now $Y(1), Y(2), \ldots$ cannot all be distinct w.p.1 unless G is continuous, and then w.p.1, $G(Y(t_0)) \neq 0$. We then have the following extension of Theorem 1, where F need not be strictly increasing.

Theorem 2. *If X and Y are stationary stochastic processes such that (3.1) and (3.2) are valid w.p.1, and $Y(1), Y(2), \ldots$ are distinct w.p.1, then for any fixed t_0, (3.4) is valid w.p.1, where F^{-1} is defined for $0 < u < 1$ by*

$$F^{-1}(u) = \sup\{x : F(x) < u\} \ . \tag{3.7}$$

If, in addition, every $X(t)$ is a bounded random variable (i.e., $F(a) = 0$ and $F(b) = 1$ for some $-\infty < a < b < \infty$), then for any fixed $t_1, \ldots, t_n$,

$$\lim_{T\to\infty} E\left(Z^T(t_1)\cdots Z^T(t_n)\right) = E\left(\tilde{Y}(t_1)\cdots\tilde{Y}(t_n)\right) \ , \tag{3.8}$$

where $\tilde{Y}(t)$ is defined on the right side of (3.4), and thus for any fixed t, τ

$$\lim_{T\to\infty} \mathrm{Cov}\left(Z^T(t), Z^T(t+\tau)\right) = \mathrm{Cov}\left(\tilde{Y}(t), \tilde{Y}(t+\tau)\right) \ . \tag{3.9}$$

Proof. The derivation of (3.4) in this case requires verifying, analogously to (3.6), that if $W_T/T \to G(Y(t_0))$ w.p.1, then $X[W_T] \to F^{-1}(G(Y(t_0)))$ w.p.1, with the definition (3.7) of F^{-1}, when $u \in (0, 1)$ is such that $F(x) = u$ on some nonempty interval (x_1, x_2). This follows from the fact that w.p.1, no $X(t)$ takes values in such an interval. Then (3.8) easily follows from (3.4) because the $X(t)$'s are bounded, and (3.9) follows from the $n = 1$ and $n = 2$ cases of (3.8). □

The right side of (3.9) is the autocovariance of $\tilde{Y}$, from which its power spectrum can be computed. Moreover, (3.7) is a standard definition that guarantees that $F^{-1}(U)$ has the distribution F, where U is uniform on (0, 1).

If the hypothesis that $Y(1), Y(2), \ldots$ all be distinct is dropped, then G may not be continuous, and there is a problem in defining S_t^T because of possible ties in

the data. One solution is to break ties "at random," which we define precisely as follows.

Let $U(1), U(2), \ldots$ be random variables that are each uniformly distributed on (0, 1), mutually independent, and independent of Y. We will break any ties (and hence obtain well-defined values of S_t^T and $Z^T(t)$) among, say, $Y(t_1), \ldots, Y(t_m)$ by treating $Y(t_i)$ as less than $Y(t_j)$ if $Y(t_i) = Y(t_j)$ but $U(t_i) < U(t_j)$. Then Theorem 2 remains valid, with essentially the same proof, if we change the definition of $\tilde{Y}$ in (3.4) to

$$\tilde{Y}(t_0) = F^{-1}\left(G^*(Y(t_0), U(t_0))\right) ,$$

where $G^*(y, u)$ is defined for real y and $u \in (0, 1)$ as

$$G^*(y, u) = G(y-) + u(G(y) - G(y-)) .$$

If y is a continuity point of G, then $G^*(y, u) = G(y)$. Whether or not G is continuous, $G^*(Y(t_0), U(t_0))$ is uniformly distributed on (0, 1).

Theorem 1 covers the numerical example in Section 5 (if we disregard the distinction between pseudorandom and random) because the distribution F of values of X and the distribution G of values of Y are both normal with mean 0 and variance 1, hence $F^{-1} \circ G$ is the identity. Thus $Z^T(t_0)$, as a sequence of random variables indexed by T, converges as $T \to \infty$, w.p.1, to $Y(t_0)$, so $Z^T(t_1)Z^T(t_2)$ converges almost surely to the corresponding product $Y(t_1)Y(t_2)$, and so finally the stochastic processes Y and Z have the same power spectrum.

4. Spectral mimicry for a nonlinear function of normal random variables

Numerical results not shown here suggest that the power spectral density of Z will sometimes inherit the redness or blueness (dominance of low or high frequencies, respectively) of the power spectral density of Y when $F^{-1} \circ G$ is not the identity. We demonstrate why that should be so in an example to illustrate computations that could be carried out for other cases.

Let $\{W(t) : t = 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) normal random variables with mean 0 and variance 1 ("standard" normal). For fixed $\delta \neq 0$, let $X(t) = W(t) + \delta(W(t))^3$. Then $\{X(t) : t = 1, 2, \ldots\}$ is a stationary discrete-time process that is not normally distributed. Using $E(W^2) = 1$, $E(W^4) = 3$, $E(W^6) = 15$, X has variance $\mathrm{Var}(X(t)) = 1 + 6\delta + 15\delta^2$. Because $\{X(t) : t = 1, 2, \ldots\}$ are i.i.d., the power spectral density of X is flat, i.e., X is a white noise.

Suppose that $\{Y(t) : t = 1, 2, \ldots\}$ is a stationary discrete-time Markov-Gaussian process, i.e., for some $\rho \in (-1, +1)$ and for all $t > 1$, $Y(t) = \rho Y(t-1) + (1 - \rho^2)^{1/2}\varepsilon(t)$, where $\varepsilon(t)$ and $Y(1)$ are i.i.d. standard normal. (In [4, pp. 418–419], Kendall and Stuart analyze this case and use this name for it.) Then $Y(t)$ is standard normal for all t. The autocorrelation and autocovariance between $Y(t)$

and $Y(t+j)$ are $\rho^{|j|}$. For frequency $\alpha \in [0, \pi]$, the spectral density of Y is

$$w(\alpha) \equiv \sum_{j=-\infty}^{\infty} \rho^{|j|} e^{i\alpha j} = \frac{1-\rho^2}{1-2\rho\cos\alpha+\rho^2} . \tag{4.1}$$

As the frequency α increases, the spectral density decreases when $\rho > 0$ and increases when $\rho < 0$, corresponding to red noise (dominated by low frequencies) and blue noise (dominated by high frequencies), respectively.

Let Z be constructed by the method of spectral mimicry, replacing each element of Y by the corresponding *order statistic* of X. Then by Theorem 1, for any fixed t_0, as $T \to \infty$, $Z^T(t_0) \to F^{-1}(G(Y(t_0)))$.

Theorem 3. *With X, Y, and Z as just defined, and with $\delta > 0$, we have, for any integer j, in the large-sample limit as $T \to \infty$:*

$$Z(t) = Y(t) + \delta(Y(t))^3 , \tag{4.2}$$

$$\mathrm{Cov}(Z(t), Z(t+j)) = (1+3\delta)^2\rho^{|j|} + 6\delta^2\rho^{3|j|} , \tag{4.3}$$

and the spectral density of Z is

$$\begin{aligned} w(\alpha) &= \frac{1}{\mathrm{Var}(Z)} \sum_{j=-\infty}^{\infty} \mathrm{Cov}(Z(t), Z(t+j)) e^{i\alpha j} \\ &= \left[(1+3\delta)^2 \frac{1-\rho^2}{1-2\rho\cos\alpha+\rho^2} \right. \\ &\qquad \left. + 6\delta^2 \frac{1-\rho^6}{1-2\rho^3\cos\alpha+\rho^6} \right] / (1+6\delta+15\delta^2) . \end{aligned} \tag{4.4}$$

Like the spectral density of Y, this spectral density of Z decreases (with increasing frequency α) when $\rho > 0$ and increases when $\rho < 0$.

Proof. To prove (4.2), i.e., that $Z(t) = H(Y(t))$ with $H(y) = y + \delta y^3$, we must show that $F^{-1}(G(y)) = H(y)$ or equivalently that $G(y) = F(H(y))$. Now $G(y) = P(Y(t) \le y) = P(W(t) \le y)$ and $F(x) = P(X(t) \le x) = P(H(W(t)) \le x)$ so that $F(H(y)) = P(H(W(t)) \le H(y)) = P(W(t) \le y)$ as desired. The last equality uses the fact that for $\delta > 0$, H is an increasing function.

Now $\mathrm{Cov}(Z(t), Z(t+j)) = \mathrm{Cov}(Y(t)+\delta(Y(t))^3, Y(t+j)+\delta(Y(t+j))^3)$. To simplify the notation, we fix t and j and write $A = Y(t)$, $B = Y(t+j)$. Then A and B are standard normal with $\mathrm{Cov}(A, B) = \rho^{|j|} \equiv \theta$, and there exists a third standard normal random variable C such that $A = \theta B + \varphi C$, where $\theta^2 + \varphi^2 = 1$ and $\mathrm{Cov}(B, C) = 0$. Hence for any functions f, g, $\mathrm{Cov}(f(B), g(C)) = 0$ and $\mathrm{Cov}(f(B), g(B)\cdot C) = E(f(B)g(B))\cdot E(C) - E(f(B))E(g(B))E(C) = 0$. Then

$$\begin{aligned} A + \delta A^3 &= (\theta B + \varphi C) + \delta(\theta B + \varphi C)^3 \\ &= \theta B + \delta\theta^3 B^3 + 3\delta\theta^2\varphi B^2 C + 3\delta\theta\varphi^2 B C^2 + \left[\varphi C + \delta\varphi^3 C^3\right] \end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Cov}\left(A+\delta A^3, B+\delta B^3\right) \\
&\quad = \mathrm{Cov}\left(B+\delta B^3, \theta B+\delta\theta^3 B^3\right)+0+E\left((B+\delta B^3)3\delta\theta\varphi^2 B\right)+0 \\
&\quad = E\left((B+\delta B^3)(\theta B+\delta\theta^3 B^3)\right)+3\delta\theta\varphi^2\cdot 1+3\delta^2\theta\varphi^2\cdot 3 \\
&\quad = \theta\cdot 1+\left(\delta\theta+\delta\theta^3\right)\cdot 3+\delta^2\theta^3\cdot 15+3\delta\theta\varphi^2+9\delta^2\theta\varphi^2 \\
&\quad = \theta+6\delta\theta+\delta^2\left(9\theta+6\theta^3\right)=(1+3\delta)^2\theta+6\delta^2\theta^3 .
\end{aligned}$$

This proves (4.3). Then (4.4) is immediate. □

Theorem 3 can be generalized to any $X(t)=H(W(t))$, where

$$H(w)=\sum_{k=0}^{m} h_k w^{2k+1}, \quad h_k\geq 0 \text{ for all } k .$$

Because such an H is an increasing function, we again have that $F^{-1}(G(Y(t)))=H(Y(t))$. Furthermore, for such an H,

$$\mathrm{Cov}(H(Y(t)), H(Y(t+j)))=E(H(Y(t))\cdot H(Y(t+j)))=\tilde{H}\left(\rho^{|j|}\right)$$

with

$$\tilde{H}(\theta)=\sum_{k=0}^{\tilde{m}} \tilde{h_k}\theta^{2k+1}, \quad \tilde{h_k}\geq 0 \text{ for all } k .$$

To see this, let A, B be as in the proof of Theorem 3. Then from the identity

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\frac{a^i}{i!}\frac{b^l}{l!}E\left(A^iB^l\right) &= E\left(e^{aA+bB}\right)=E\left(e^{a(\theta B+\varphi C)+bB}\right) \\
&= e^{(a\theta+b)^2/2+(a\varphi)^2/2}=e^{ab\theta+a^2/2+b^2/2}
\end{aligned}$$

it follows that $E\left(A^{2k+1}B^{2k'+1}\right)$ is a polynomial in odd powers of θ with nonnegative coefficients. The step from (4.3) to (4.4) remains valid when the right side of (4.3) is any polynomial in $\rho^{|j|}$. Provided that the polynomial contains only odd powers of $\rho^{|j|}$ and that no coefficient of these odd powers is negative, it follows that the power spectral density of Z monotonically decreases (or increases) with increasing frequency if the power spectral density of Y decreases (or increases) with increasing frequency.

5. Numerical example

To illustrate the method, we give numerical examples using MATLAB (version 4.2c on a Unix workstation), first with $T=5$, then with $T=1024$. In this version

of MATLAB, a call to `randn` produces a normally distributed pseudorandom value with mean 0 and standard deviation 1. The number returned from each call to `randn` is supposed to be statistically independent of the result from any other call to `randn`.

With $T = 5$, five calls to `randn` produced the time series X : $X(1) = 1.0164 = X[5]$, $X(2) = 0.4995 = X[4]$, $X(3) = -0.7754 = X[2]$, $X(4) = -1.3084 = X[1]$, $X(5) = -0.0475 = X[3]$. In principle, a time series (sequence of random variables, not necessarily these particular data) defined in this way has a white power spectrum because successive elements are statistically independent, and therefore all frequencies are present with equal power in the power spectrum.

To produce a time series Y with a reddened spectrum, we began by setting $Y(1) =$ `randn` as before (as each call to `randn` is independent, $Y(1)$ and $X(1)$ need not be the same), then computed $Y(t) = 0.9$*$Y(t-1) + 0.4359$*`randn` for $t = 2, 3, 4, 5$. Because $(0.9)^2 + (0.4359)^2 = 1$, $Y(t)$ is normally distributed with mean 0 and standard deviation 1 for each t, and Y is stationary. The result was: $Y(1) = -0.6658 = Y[4]$, $Y(2) = -0.6379 = Y[5]$, $Y(3) = -0.7746 = Y[2]$, $Y(4) = -0.8109 = Y[1]$, $Y(5) = -0.6927 = Y[3]$. In principle, the positive serial autocorrelation among elements of Y reddens the power spectrum, that is, gives greater power at lower frequency. Here the functions F and G of Theorem 1 would each be the standard normal distribution function.

To obtain a reddened series Z with exactly the values of X and the approximate power spectrum of Y, we replaced each value of Y with the corresponding order statistic from X : $Z(1) = X[4] = 0.4995$, $Z(2) = X[5] = 1.0164$, $Z(3) = X[2] = -0.7754$, $Z(4) = X[1] = -1.3084$, $Z(5) = X[3] = -0.0475$.

In this example, $X(1), \ldots, X(5)$ and $Y(1), \ldots, Y(5)$ are taken from a longer time series for X and Y with $T = 1024$. To illustrate the method and results with time series data of meaningful length, Figure 1 plots X and its power spectrum, obtained with the fast Fourier transform. (The power spectrum plotted shows the logarithm of the squared modulus or squared absolute value of elements 2 to 512 of the 1024-element fast Fourier transform produced by the MATLAB function `fft`.) When a least-squares straight line is fitted to the plotted log-power spectrum as a function of the logarithm of the frequency (which runs from 1/1022 to 1/2), the result is log(power) = -0.0048*log(index) + 6.3477. Here and later, the index runs from 1 to 511. The coefficient -0.0048 has a 95% confidence interval $(-0.1201, +0.1106)$, which includes zero, and $P = 0.9354$ for the null hypothesis that the slope is 0. (STEPWISE in the MATLAB Statistics Toolbox was used for the statistical computations [3].) By visual inspection, the power spectrum is approximately flat, hence X is approximately white.

Figure 2 plots Y and its power spectrum. The values on the vertical axis of the power spectra (though not the time series) differ from those in Figure 1; the power is distributed differently over the frequencies. By least-squares, log(power) = -1.4994*log(index) + 2.2072. The coefficient -1.499 has a 95% confidence interval $(-1.616, -1.383)$, which excludes zero, as theory predicts. The power spectrum declines with increasing frequency, hence Y is reddened.

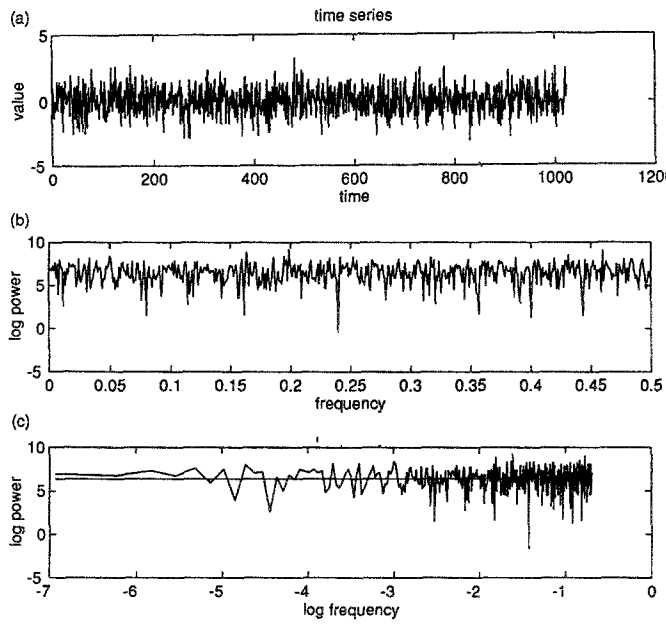

Figure 1. (a) A pseudo-white-noise time series X of length $T = 1024$; (b) its power spectrum as a function of frequency; (c) its power spectrum as a function of log-frequency, displayed together with a fitted least-squares line.

Figure 3 plots Z and its power spectrum. The time series values of Z are identical to those of X, but the power spectrum declines with increasing frequency like the power spectrum of Y. By least-squares, log(power) $= -1.4705$*log(index) $+$ 2.3142. The coefficient -1.471 has a 95% confidence interval $(-1.581, -1.360)$, which excludes zero and essentially coincides with the confidence interval of the slope coefficient for Y. Thus the slope coefficients of Y and Z do not differ statistically.

6. Generalizations

The method described here is potentially widely applicable. It is not restricted to generating pairs of time series with identical elements and different spectra: the same method could be used to obtain a trio of time series, one with a red, one with a white, and one with a blue spectrum, and all with the same elements permuted in different orders. The method could also be applied to signals that vary in space as well as in time. More generally, observations or data points could be indexed by a parameter that is multidimensional rather than single dimensional, as in the case of time.

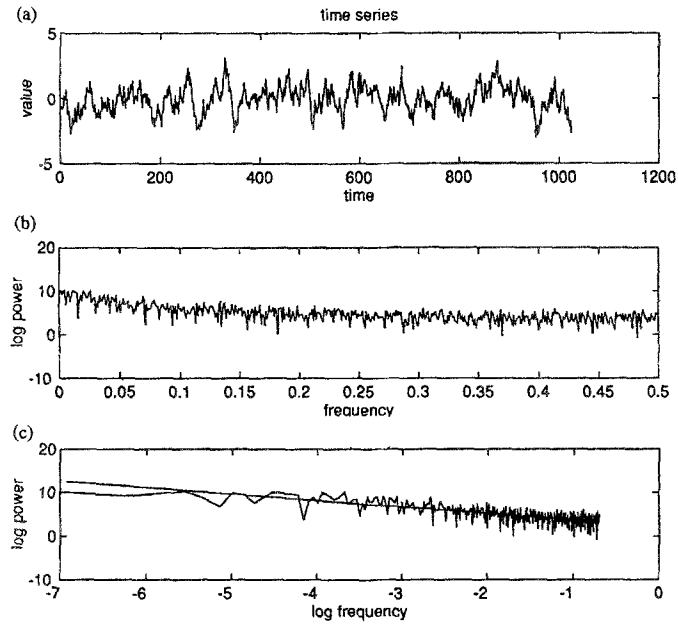

Figure 2. (a) A pseudo-red-noise time series Y of length $T = 1024$; (b) its power spectrum as a function of frequency; (c) its power spectrum as a function of log-frequency, displayed together with a fitted least-squares line.

The method does not require that the set of values $\{X(t)\}$ be derived from a time series, because only the empirical distribution of X is used. It is sufficient to select order statistics from an arbitrary CDF or to generate values in some other way.

The method could be used to mimic other characteristics that depend on the order of data, in addition to the power spectrum as illustrated here. As one example, the method could mimic a sine wave of one frequency with a sine wave of another frequency using identical elements permuted in a different order. As another example, the method could be used to generate a time series Z, the first differences of which are identical to the first differences of the given time series X, where the order statistics of the first differences of Z occur in the same order as the order statistics of the first differences of Y. MATLAB [5] provides a first-difference function `diff`, so all that would be required is to compute the differences of Z by `diffz = mimicry(diff(x), diff(y))` and then to invert the differencing of Z by the cumulative-sum command `z = cumsum([z0; diffz])`, where the initial value `z0` is arbitrary and may be chosen to give `z` any desired location. The MATLAB function `mimicry` is described in the Appendix. More generally, if `f` is

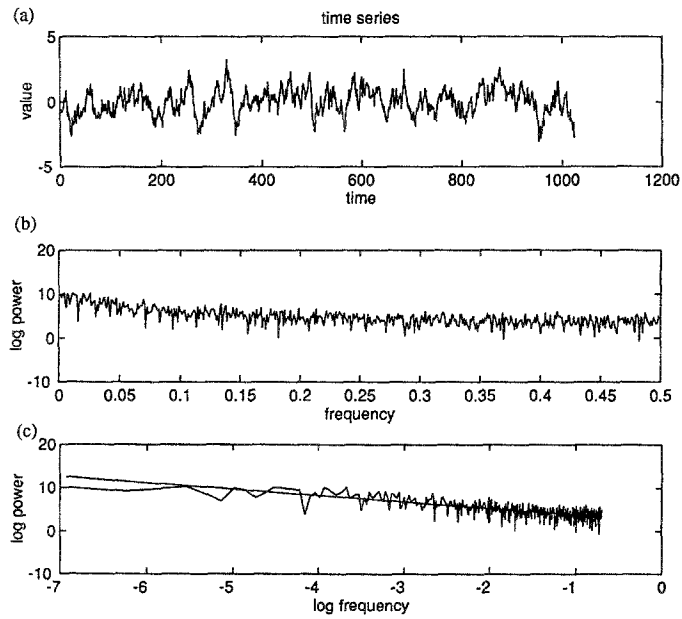

Figure 3. (a) A pseudo-red noise time series Z with values identical to those in X; (b) its power spectrum as a function of frequency; (c) its power spectrum as a function of log-frequency, displayed together with a fitted least-squares line.

an invertible function with inverse function g, one can compute a series Z whose elements are those of $f(X)$ (such as the differences of X in the previous example) arranged according to the order statistics of $f(Y)$ by means of `g(mimicry(f(x), f(y))`.

Other areas of science besides ecology could exploit this method. For example, it could be used to examine responses to varying spectra in other biological systems, such as individual nerve cells, nervous tissues, and the auditory or other organ systems.

Appendix: MATLAB program

A simple program or function in MATLAB [5] carries out spectral mimicry. This function depends on the MATLAB-provided function `sort`, which works as follows. If `x` is a vector of real numbers, the statement `[xsort, xindex] = sort(x)` places in the vector `xsort` the elements of `x` arranged in ascending order. The vector `xindex` gives the indices of the elements in `x` in the sorted order; that is, `xsort = x(xindex)`. For example, if `x = (4, 3)`, then `xsort = (3, 4)`, `xindex = (2, 1)`.

```
function z = mimicry (x,y)
% z = mimicry(x,y)
% x, y: 2 real vectors of length T
% z: real vector of length T in which the elements of x
%     occur in the same rank order as the elements of y
% 10 September 1995
[xsort,xindex] = sort(x);
[ysort,yindex] = sort(y);
[zsort,zindex] = sort(yindex);
z = xsort(zindex);
return
```

That there is a 1 to 1 correspondence between elements of X and elements of Z is clear from this code because `xsort` is a permutation of the elements of `x`, and `z` is a permutation of the elements of `xsort` and therefore also a permutation of the elements of `x`.

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